Space guaranteeing a primitive chaotic behavior

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Abstract. This study describes such a situation that a Cantor set emerges as a result of the exploration of sufficient conditions for the property which is generalized from fundamental chaotic maps, and the Cantor set even guarantees infinitely many varieties of the behavior with the property, as well as a typical continuum.

1 Introduction

We are surrounded by chaotic behaviors and complex forms called fractals, and it is known that a fractal emerges in a chaotic behavior [1, 2, 3, 4, 5]. The purpose of this study is to reveal the relation of a fractal and a chaos from a new angle.

Fundamental maps in the chaos theory, such as the tent map $\varphi : [0,1] \to [0,1]$, $x \mapsto \min\{2x, 2(1-x)\}$, the baker map $\varphi : [0,1] \to [0,1]$, $x \mapsto 2x$ $(0 \le x \le 1/2)$, $2x - 1$ $(1/2 \le x \le 1)$, and the logistic function $\varphi : [0,1] \to [0,1]$, $x \mapsto 4x(1-x)$, have an intriguing property

> "For any infinite sequence ω_0, ω_1, $\omega_2, \ldots$, there exists an initial point $x_0 \in \omega_0$ such that $\varphi(x_0) \in \omega_1$, $\varphi(\varphi(x_0)) \in \omega_2, \ldots$, where each ω_i is $[0, 1/2]$ or $[1/2, 1]$."

This property can play an important role in the guarantee of nonperiodicity, sensitive dependence on initial conditions, and so on for the maps [4]. Then, it was pointed out [6] that this property implies general problems about determinism, causality, free will, and so on, which are vital problems in sciences (e.g. Ref. [7, 8, 9, 10, 11, 12, 13]).

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However, this property is too restricted in such a sense that the number of the subsets $[0, 1/2]$ and $[1/2, 1]$ which imply events or selections [6] is only two and they are required for intersecting one point $1/2$. Then, the following property (P) generalized from this property was proposed as a primitive chaotic behavior, and it has been investigated [6, 14, 15, 16].

(P) For any infinite sequence ω_0, ω_1, $\omega_2, \ldots$, there exists an initial point $x_0 \in \omega_0$ such that $f_{\omega_0}(x_0) \in \omega_1$, $f_{\omega_1}(f_{\omega_0}(x_0)) \in \omega_2, \ldots$. Here, each ω_i is an element of a family $\{X_\lambda, \lambda \in \Lambda\}$ of nonempty subsets of a set X, and each f_{X_λ} is the map from X_λ to X.

In a previous study [6], such a situation that a nondegenerate Peano continuum emerges by the exploration of sufficient conditions for the property (P) was described, and the following theorem was exhibited as a result of it, where a nondegenerate space means the space that consists of more than one point, a Peano continuum is a locally connected continuum, and a continuum is a nonempty connected compact metric space.

Theorem 1 *If X is a nondegenerate Peano continuum, for any $\varepsilon > 0$, there exist finitely many infinite Peano subcontinua $X_1, \ldots, X_n$ of X covering X such that dia $X_i < \varepsilon$, $i = 1, \ldots, n$. Then, for each i, for any positive integer m^i, for any m^i points $x_1^i, \ldots, x_{m^i}^i \in X_i$ and $y_1^i, \ldots, y_{m^i}^i \in X$, there exists a continuous onto map $f_{X_i} : X_i \to X$ such that $f_{X_i}(x_1^i) = y_1^i, \ldots, f_{X_i}(x_{m^i}^i) = y_{m^i}^i$, and the space X, the subsets of X, $X_1, \ldots, X_n$, and the maps $f_{X_1}, \ldots, f_{X_n}$ satisfy the property (P).*

The nondegenerate Peano continuum such as n-cells, n-spheres, dendrites, spaces homeomorphic to a torus, spaces homeomorphic to a solid torus, and so on [17] is ordinarily observed in the real world, phase spaces, and so on, which is both the representation of diverse actual matters and the representation for the realization of various natural phenomena [18, 19]. Then, the relation between the nondegenerate Peano continuum and properties such as hierarchy, self-similarity, and so on was pointed out [16].

In addition, the nondegenerate Peano continuum guarantees infinitely many varieties of the behavior with the property (P) owing to the arbitrariness of the positive number ε of Theorem 1. The above fundamental chaotic maps are simple examples of maps guaranteed from this theorem.

In this study, let us make a new exploration of sufficient conditions for the property (P).

2 Sufficient conditions for the property (P)

Let us start by recalling the following proposition [6], where a topological space X is said to be countably compact provided that every countable open cover of X has a finite subcover.

Proposition 1 *If X is a countably compact space, $\{X_\lambda, \ \lambda \in \Lambda\}$ is a family of nonempty closed subsets of X, and each f_{X_λ} is a continuous map from X_λ onto X, they satisfy the property* (P).

From this proposition, the existence of the behavior with the property (P) is guaranteed by the existence of such a space X and families $\{X_\lambda, \ \lambda \in \Lambda\}$ and $\{f_{X_\lambda}, \ \lambda \in \Lambda\}$.

However, as the existence of the continuous surjections seems to be artificial than the other conditions [6] or more essential for the property (P), let us prepare the following proposition which is verified in Appendix for the purpose of the guarantee of this existence, where a topological space Y is perfect provided that Y contains no isolated points, a topological space Y is zero-dimensional provided that there is a base for the topology of Y such that each element of the base is a closed and open subset of Y, and a topological space Y is a T_1-space provided that for each $y \in Y$, $\{y\}$ is a closed subset of Y.

Proposition 2 *Let A be a zero-dimensional perfect compact T_1-space. For any compact metric space X, there exists a continuous map from A onto X.*

From this proposition, if X is a compact metric space and each X_λ is a nonempty zero-dimensional perfect closed subset of X, there exist continuous surjections $f_{X_\lambda} : X_\lambda \to X, \ \lambda \in \Lambda$, and thus they satisfy the property (P) from Proposition 1. Namely, the existence of such a space X and a family $\{X_\lambda, \ \lambda \in \Lambda\}$ guarantees the existence of the behavior with the property (P).

However, this time, as the conditions of the family $\{X_\lambda, \ \lambda \in \Lambda\}$ of Proposition 2 seems to be artificial or essential, let us further prepare the following proposition for the guarantee of the existence, where a T_0-space is a topological space Y such that for any points y_1 and y_2 in Y with $y_1 \neq y_2$, there exists an open subset U of Y such that $y_i \in U$ and $y_j \notin U$ for some choice of i and j.

Proposition 3 *Let X be a zero-dimensional perfect T_0-space. For any positive integer n, there exist nonempty closed and open subsets $X_1, \ldots, X_n$ of X such that $X_i \cap X_{i'} = \emptyset$, $i \neq i'$, $X_1 \cup \cdots \cup X_n = X$; that is, $\{X_1, \ldots, X_n\}$ is a partition of X each element of which is a closed and open subset of X.*

Proof Let us use the mathematical induction. Let $\{X_1, \ldots, X_{n-1}\}$ be a partition of X each element of which is a closed and open subset of X. From the perfectness of X, X_{n-1} contains two different points x and y, because if X_{n-1} is a singleton $\{x\}$, x is an isolated point in X.

Without loss of generality, there exists an open subset u of X such that $x \in u$ and $y \notin u$, because X is a T_0-space. Since X is zero-dimensional, there exists a closed and open subset v of X such that $x \in v \subset u \cap X_{n-1}$. Since $X_{n-1} - v$ contains y and is a closed and open subset of X, the desired partition $\{X_1, \ldots, X_{n-2}, X_{n-1} - v, v\}$ is obtained. $\square$

Here, any nonempty open subset A of a perfect space Y is perfect, because if a is an isolated point in A, a is also an isolated point in Y. Then, any nonempty subset A of a zero-dimensional space Y is zero-dimensional. This is because if $\{U_\lambda, \ \lambda \in \Lambda\}$ is a base for the topology of Y each element of which is a closed and open subset of Y, $\{U_\lambda \cap A, \ \lambda \in \Lambda\}$ is a base for the topology of the subspace A of Y each element of which is a closed and open subset of A.

Accordingly, if X is a zero-dimensional perfect compact metric space, for any positive integer n, there exists a partition $\{X_1, \ldots, X_n\}$ of X each element of which is a zero-dimensional perfect closed subsets of X. From Proposition 2, there exist continuous surjections $f_{X_i} : X_i \to X$, $i = 1, \ldots, n$, and the existence of the behavior with the property (P) is guaranteed from Proposition 1.

Furthermore, lecalling that a space is a Cantor set if and only if it is a zero-dimensional perfect compact space [20, p.65], where any space that is homeomorphic to the Cantor middle-third set is called a Cantor set, we obtain the following theorem.

Theorem 2 *If X is a Cantor set, for any positive integer n, there exist a partition $\{X_1, \ldots, X_n\}$ of X each element of which is a closed and open subset of X and continuous surjections $f_{X_i} : X_i \to X$, $i = 1, \ldots, n$. Then, they satisfy the property (P).*

In addition, the Cantor set guarantees infinitely many varieties of the behavior with the property (P), as well as the nondegenerate Peano continuum, owing to the arbitrariness of the positive integer n of this theorem. In particular, the map $f : S \to S$, $x \mapsto 3x$ $(x \in [0, 1/3] \cap S)$, $3x - 2$ $(x \in [2/3, 1] \cap S)$ is a simple example of maps guaranteed from this theorem where S denotes the Cantor middle-third set, and the property (P) can play an important role in the guarantee of nonperiodicity, sensitive dependence on initial conditions, and so on for this map.

3   Conclusions

It is exhibited that a Cantor set homeomorphic to the Cantor set which is a fundamental fractal emerges as a result of the exploration of sufficient conditions for the property (P) generalized from a property specific to fundamental chaotic maps, and the Cantor set even guarantees infinitely many varieties of the behavior with the property (P).

It is interesting that the contrast spaces, the Cantor set and the nondegenerate Peano continuum, emerge by the exploration of the same property (P); the Cantor set is totally disconnected [17, Theorem 7.14] while the nondegenerate Peano continuum is connected and locally connected by definition.

Acknowledgments

A   Proof of Proposition 2

Since X is a compact metric space, there exist finitely many open balls $S_1, \ldots, S_n$ such that $S_1 \cup \cdots \cup S_n = X$ and $dia\, S_i$ (the diameter of S_i) $< 1/2$ for each i. Letting X_i be the closure of S_i in X for each i, we obtain nonempty closed subsets $X_1, \ldots, X_n$ of X such that $X_1 \cup \cdots \cup X_n = X$ and $dia\, X_i < 1$ for each i.

From Proposition 3, there exist a partition $\{A_1, \ldots, A_n\}$ of A each element of which is a closed and open subset of A. Then, the map

$$g_1 : A \to \Im, \ a \mapsto X_i \ (a \in A_i) \tag{1}$$

is obtained, where $\Im$ denotes the family of all closed subsets of X.

Since each subspace X_i of X is a compact metric space, there exist finitely many open balls $S_{i1}, \ldots, S_{in_i}$ of X_i such that $S_{i1} \cup \cdots \cup S_{in_i} = X_i$ and $dia\, S_{ij} < 1/4$ for each j. Letting X_{ij} be the closure of S_{ij} in X_i for each j, we obtain nonempty closed subsets $X_{i1}, \ldots, X_{in_i}$ of X such that $X_{i1} \cup \cdots \cup X_{in_i} = X_i$ and $dia\, X_{ij} < 1/2$.

From Proposition 3, for each i, there exist a partition $\{A_{i1}, \ldots, A_{in_i}\}$ of A_i each element of which is a closed and open subset of A. Then, the map

$$g_2 : A \to \Im, \ a \mapsto X_{ij} \ (a \in A_{ij}) \tag{2}$$

is obtained.

Since this procedure can be repeated, the family $\{g_k : A \to \Im, \ k \in \mathbb{N}\}$ of the surjections such that for each k,

$$dia\, g_k(a) < \frac{1}{k}, \ ^\forall a \in A, \tag{3}$$

is obtained. Since $\bigcap_{k=1}^{\infty} g_k(a)$ is a singleton for each $a \in A$, the map

$$G : A \to \{\{x\}; \ x \in X\}, \ a \mapsto \bigcap_{k=1}^{\infty} g_k(a) \tag{4}$$

is obtained.

Next, let us verify that the map

$$p \circ G : A \to X \tag{5}$$

is continuous, where

$$p : \{\{x\}; \ x \in X\} \to X, \ \{x\} \mapsto x. \tag{6}$$

For any $a \in A$ and for any open subset U of X such that $p \circ G(a) \in U$, since $\bigcap_{k=1}^{\infty} g_k(a) \subset U$, there exists k_0 such that $g_{k_0}(a) \subset U$ from Lemma 2. By the definition of g_{k_0}, there exists an open (and closed) subset u of A such that

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$a \in u$ and $g_{k_0}(a') \subset U$ for any $a' \in u$. Since $G(a') \subset g_{k_0}(a') \subset U$ for any $a' \in u$, $p \circ G(u) \subset U$, and thus $p \circ G$ is continuous.

Lastly, let us verify that $p \circ G$ is a surjection. For any $x \in X$, there exists $a_k \in A$ such that $x \in g_k(a_k)$ for each k, by the definition of g_k. Accordingly, the sequence $\{a_k\}_{k=1}^{\infty}$ of points in A such that $x \in g_k(a_k)$ for each k, is obtained.

Let us consider a case such that $\{a_k\}_{k=1}^{\infty}$ is a finite set. Since there exists a number K such that $a_K = a_k$ for any $k \geq K$, $G(a_K) = \bigcap_{k=1}^{\infty} g_k(a_K) = \{x\}$. Namely, $p \circ G(a_K) = x$.

Let us consider a case such that $\{a_k\}_{k=1}^{\infty}$ is an infinite set. Since A is compact, there exists an accumulation point a of $\{a_k\}_{k=1}^{\infty}$. If $G(a) \neq \{x\}$, $G(a) \subset X - \{x\}$, and thus there exists k_0 such that $g_{k_0}(a) \subset X - \{x\}$ from Lemma 2. Accordingly, there exists an open (and closed) subset u of A such that $a \in u$ and $g_{k_0}(a') \subset X - \{x\}$ for any $a' \in u$. From Lemma 1, there exists an open subset u' of A such that $a \in u' \subset u$ and $u' \cap (\{a_k\}_{k=1}^{\infty} - \{a_1, \ldots, a_{k_0}\}) \neq \emptyset$. Therefore, there exists $a_{k'} \in u'$ $(k' > k_0)$, and thus $x \in g_{k'}(a_{k'}) \subset g_{k_0}(a_{k'}) \subset X - \{x\}$. This is a contradiction, and thus $G(a) = \{x\}$. Namely, $p \circ G$ is a surjection. $\square$

Lemma 1 *Let Y be a T_1-space, A be an infinite subset of Y, and a be an accumulation point of A. For any finite subset B of A and for any open subset U of Y such that $a \in U$, there exists an open subset u of Y such that $a \in u \subset U$ and $u \cap (A - B) \neq \emptyset$.*

Proof Since Y is a T_1-space, there exists an open subset U' of Y such that $a \in U'$ and $U' \cap B = \emptyset$. Since a is an accumulation point of A, $(U \cap U') \cap A \neq \emptyset$, and thus $(U \cap U') \cap (A - B) \neq \emptyset$. $\square$

Lemma 2 *Let Y be a compact T_1-space and $\{A_i\}$ be the sequence of nonempty closed subsets of Y such that $A_i \supset A_{i+1}$ for each i. For any open subset U of Y such that $\bigcap_{i=1}^{\infty} A_i \subset U$, there exists i_0 such that $A_{i_0} \subset U$.*

Proof If there exists an open subset U of Y such that $\bigcap_{i=1}^{\infty} A_i \subset U$ and $A_i \not\subset U$ for any i, there exists a sequence $\{a_i\}_{i=1}^{\infty}$ of points in A such that $a_i \in A_i \cap (X - U)$ for each i.

Let us consider a case such that $\{a_i\}_{i=1}^{\infty}$ is a finite set. Since there exists a number I such that $a_I = a_i$ for any $i \geq I$, $a_I \in \bigcap_i A_i \subset U$. This is a contradiction.

Let us consider a case such that $\{a_i\}$ is an infinite set. Since X is compact, there exists an accumulation point a of $\{a_i\}_{i=1}^{\infty}$. If $a \in U$, there exists i' such that $a_{i'} \in U$. This is a contradiction. Thus,

$$a \in X - U. \tag{7}$$

Let us verify that a is contained in the closure of $A_{i'}$ for each i'. For any open subset u of Y such that $a \in u$, there exists an open subset u' of Y such that $a \in u' \subset u$ and $u' \cap (\{a_i\}_{i=1}^{\infty} - \{a_1, \ldots, a_{i'}\}) \neq \emptyset$, from Lemma 1. Accordingly, $u \cap A_{i'} \supset u' \cap (\{a_i\}_{i=1}^{\infty} - \{a_1, \ldots, a_{i'}\}) \neq \emptyset$, and thus a is contained in the closure of $A_{i'}$.

Since each $A_{i'}$ is closed, $a \in \bigcap_{i'} A_{i'} \subset U$. This is in contradiction with the relation (7). $\square$

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